Understanding Polysign Numbers the Standard Way

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Abstract

For several years, the notion of *polysign numbers* has been advocated in usenet discussions by T. Golden e.g. as a new basis for theoretical physics. This article will show that these structures are not new after all, but rather well-known to algebraists. The facts listed here are not new either and have already been used in responses by several people in the discussions mentioned, i.e. the author does not claim that any of them be attributed to him.

1 Polysign Numbers

The inventor of polysign numbers[1], T. Golden, views magnitude as fundamental and wants to extend the two signs + and − known from traditional mathematics by using three, four or more signs.[2] The various polysign structures obtained this way are denoted as $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and so on, depending on the number of signs used (there is no intention to define $\mathbf{P}_0$). In traditional math, we have the equality

$$-x + x = 0$$

from the very definition of $-x$ as additive inverse of x. This is taken as a motivation for a relation between the two signs. That is, in $\mathbf{P}_2$, which is built from *magnitudes* (non-negative numbers) with two signs − and +, we have the identity ("cancellation")

$$-x{+}x = 0.$$

For $\mathbf{P}_3$, a third sign $\star$ is summoned and the identity

$$-x{+}x{\star}x = 0$$

postulated instead, for $\mathbf{P}_4$ a fourth sign # and the identity

$$-x{+}x{\star}x\#x = 0.$$

In general, $\mathbf{P}_n$ uses n signs denoted $\sigma_1, \dots, \sigma_n$ and the cancellation identity becomes

$$\sum_{i=1}^{n} \sigma_i x = 0. \tag{1}$$

[1] http://bandtechnology.com/PolySigned/

[2] To distinguish polysign signs from traditional mathematical operator symbols, the former will always be shown in red in this paper.

1.1 $\mathbf{P}_1$

The following discussion will discuss $\mathbf{P}_n$ for $n \geq 1$, but we will not lay emphasis on the case $n = 1$. The reason for this is that $\mathbf{P}_1$ is not as clearly defined as one might hope—or rather there are contradictory interpretations of the definition. For one, T. GOLDEN imposes an order on the sign symbols by which the first sign is always $-$, hence the elements of $\mathbf{P}_1$ are of the form $-x$ and not $+x$ as perhaps expected and sometimes implicitly used. Moreover, the cancellation identity in $\mathbf{P}_1$ states that $-x = 0$ holds for all x, thus $\mathbf{P}_1$ consists of only one point (and not of the set of non-negative reals as sometimes stated).

Of course, the investigations below will also establish this observation that $\mathbf{P}_1$ is a one-element set.

1.2 Addition

Additon of polysign numbers is defined componentwise, i.e. if x_i, y_i are magnitudes ($i = 1, \ldots, n$), then

$$\sum_{i=1}^{n} \sigma_i x_i + \sum_{i=1}^{n} \sigma_i y_i := \sum_{i=1}^{n} \sigma_i (x_i + y_i).$$

For example, in $\mathbf{P}_4$ ths means that

$$(-x_1 + x_2 \star x_3 \# x_4) + (-y_1 + y_2 \star y_3 \# y_4) = -(x_1+y_1) + (x_2+y_2) \star (x_3+y_3) \# (x_4+y_4).$$

It is straightforrwad that this is well-defined—adding a constant to all x_i and/or all y_i also adds a constant to all components of the result, hence leaves it unchanged by virtue of the cancellation identity.

1.3 Multiplication

To define multiplication of polysign numbers, it is sufficient to define all products of signs and then extend this definition in the obvious way (i.e. linearly). The simple rule is that $(\sigma_i 1) \cdot (\sigma_j 1) = \sigma_k 1$ if $i + j \equiv k \pmod{n}$. In $\mathbf{P}_2$, this yields the basic rules so hard to capture by beginners, namely

$$\begin{aligned}
(-1) \cdot (-1) &= +1 \\
(-1) \cdot (+1) &= -1 \\
(+1) \cdot (-1) &= -1 \\
(+1) \cdot (+1) &= +1.
\end{aligned}$$

When moving to e.g. $\mathbf{P}_4$, this list becomes much longer (16 entries) and while we still have $(-1) \cdot (-1) = +1$ in $\mathbf{P}_4$, we must acquaint ourselves with $(+1) \cdot (+1) = \#1$.

In general

$$\left(\sum_{i=1}^{n} \sigma_i x_i \right) \cdot \left(\sum_{i=1}^{n} \sigma_i y_i \right) = \sum_{k=1}^{n} \sigma_k \sum_{\substack{1 \leq i,j \leq n \\ i+j=k \vee i+j=k+n}} x_i y_j.$$

It is left as an exercise to show that this is well-defined with respect to (1) and that the distributive law holds for addition and multiplication as defined here. Doing so is however hardly necessary as we are about to establish an isomorphism with simply defined rings, thus immediately showing that $\mathbf{P}_n$ is also a ring.

2 An Algebra isomorphic to $\mathbf{P}_n$

It is clear that polysign numbers can be multiplied by non-negative real numbers (i.e. magnitudes) componentwise and because of (1) all polysign numbers have additive inverses. For example in $\mathbf{P}_4$, we see that $y = -3+3\#2$ is inverse to $x = \star 3\#1$ and hence we write $y = -x$ (note that this does not say $y = -x$!). One can then define the map $\iota\colon \mathbb{R} \to \mathbf{P}_n, x \mapsto \sigma_4 \max\{x, 0\} - \sigma_4 \max\{-x, 0\}$ since both $\max\{x, 0\}$ and $\max\{-x, 0\}$ are valid magnitudes. A straightforward though clumsy check (one needs to distinguish cases for all signs of the elements of $\mathbb{R}$ involved) shows that ι is compatible with addition and multiplication, hence at least the image $\iota(\mathbb{R}) \subseteq \mathbf{P}_n$ is a ring. For $n > 1$ there is no evident kernel, hence we suspect $\iota(\mathbb{R})$ to be isomorphic to $\mathbb{R}$, and we will see this in a moment.

By the universal property of polynomial rings, extending the definition of ι to $\mathbb{R}[X]$ consists essentially of defining the image of X. A useful choice is to map X to -1 since all other signs can be obtained from $-$ by repeated multiplication. Thus we define $\Phi\colon \mathbb{R}[X] \to \mathbf{P}_n$ by letting $\Phi(a) = \iota(a)$ for $a \in \mathbb{R}$ and $\Phi(X) = \sigma_1 1$. This is compatible with addition and multiplication by definition, thus showing that at least $\Phi(\mathbb{R}[X]) \subseteq \mathbf{P}_n$ is a ring (and in fact an $\mathbb{R}$-algebra).

From (1) it is clear that at least the polynomial $f_n := 1 + X + \cdots + X^{n-1}$ is in $\ker \Phi$, thus we obtain an $\mathbb{R}$-algebra homomorphism

$$\overline{\Phi}\colon \mathbb{R}[X]/(f_n) \to \Phi(\mathbb{R}[X]) \subseteq \mathbf{P}_n. \tag{2}$$

Moreover, the map $\mathbf{P}_n \to \mathbb{R}[X]/(f_n)$ that maps

$$\sum_{i=1}^{n} \sigma_i x_i \mapsto \sum_{i=1}^{n} x_i X^i + (f_n)$$

is well-defined by the choice of f_n and an obvious inverse of $\overline{\Phi}$. Hence $\overline{\Phi}$ is in fact bijective and an isomorphism of $\mathbb{R}$-algebras. (Especially, ι is injective as suspected provided $n > 1$). This proves

Theorem 1. *For $n \in \mathbb{N}$ let $f_n = 1 + X + \cdots + X^{n-1} = \frac{X^n-1}{X-1} \in \mathbb{R}[X]$. Then we have that*

$$\mathbf{P}_n \cong \mathbb{R}[X]/(f_n)$$

via an $\mathbb{R}$-algebra isomorphism that sends $\sigma_k 1$ to $X^k + (f_n)$.

3 Small Values of n

For small values of n, the left hand side of (2) is readily simplified (all isomorphisms being $\mathbb{R}$-algebra isomorphisms):

$$\begin{aligned}
\mathbb{R}[X]/(f_1) &= \mathbb{R}[X]/(1) &&\cong 0, \\
\mathbb{R}[X]/(f_2) &= \mathbb{R}[X]/(1+X) &&\cong \mathbb{R}, \\
\mathbb{R}[X]/(f_3) &= \mathbb{R}[X]/(1+X+X^2) &&\cong \mathbb{C} \quad (\text{per } X \mapsto \tfrac{-1+i\sqrt{3}}{2}), \\
\mathbb{R}[X]/(f_4) &= \mathbb{R}[X]/(1+X+X^2+X^3) &&\cong \mathbb{R} \oplus \mathbb{C} \quad (\text{per } X \mapsto (-1, i)).
\end{aligned}$$

4 Direct Sums of $\mathbb{R}$ and $\mathbb{C}$ isomorphic to $\mathbf{P}_n$

The observations in section 3 generalize as follows:

Theorem 2. *Let $\zeta \in \mathbb{C}$ be a primitive nth root of unity. Then there is an $\mathbb{R}$-algebra isomorphism* See ERRATA

$$\Psi\colon \mathbf{P}_n \to \mathcal{A}_n := \begin{cases} \mathbb{C}^{m-1} \oplus \mathbb{R} & \text{if } n = 2m \text{ is even} \\ \mathbb{C}^m & \text{if } n = 2m+1 \text{ is odd} \end{cases} \tag{3}$$

that maps $\sigma_k 1 \mapsto (\zeta^k, \zeta^{2k}, \dots, \zeta^{mk})$.

Proof. Let $f = f_n$. Then the polynomial f has $n-1$ distinct roots ζ^k in $\mathbb{C}$ ($k = 1, \dots, n-1$). At most one of these roots is real (and only if n is even). Therefore f is the product of m polynomials $g_1, \dots, g_m$ that are irreducible in $\mathbb{R}[X]$ and all of degree 2, except that $\deg g_m = 1$ if $n = 2m$. More precisely, we can take $g_k(X) = X^2 - (\zeta^k + \overline{\zeta}^k)X + 1$ for $k = 1, \dots, m$ except that $g_m(X) = X+1$ if $n = 2m$. Since the g_k are pairwise coprime, the chinese remainder theorem gives us an isomorphism

$$\mathbb{R}[X]/(f) \cong \bigoplus_{k=1}^{m} \mathbb{R}[X]/(g_k)$$

that is given by sending $X + (f_n)$ to $X + (g_k)$ in the kth component. If $\deg g_k = 2$, then we have an isomorphism $\mathbb{R}[X]/(g_k) \cong \mathbb{C}$ that is given by $X + (g_k) \mapsto \zeta^k$ and in case $n = 2m$ we have $\mathbb{R}[X]/(g_m) = \mathbb{R}[X]/(X+1) \cong \mathbb{R}$ where necessarily $X + (g_m) \mapsto -1 = \zeta^m$.

Putting this together with the isomorphism described in theorem 1 we obtain an isomorphism Ψ as desired. □

5 Norm

On $\mathbb{R}$ and $\mathbb{C}$ we have the standard absolute value. It has the advantage that it is multiplicative, i.e. $|ab| = |a||b|$ holds for all a, b. It is tempting to define a norm on the algebra $\mathcal{A}_n$ occurring in (3) by $\sqrt{|z_1|^2 + \cdots + |z_m|^2}$, i.e. by viewing the algebra as standard vector space $\mathbb{R}^{n-1}$ and taking the standard EUKLIDean norm. However, this disregards the algebra structure. In order to make the unit element of the algebra have norm 1 and to better represent the interrelation between the "two halves" of the complex components, one commonly prefers the following norm: Let d_j be the dimension of the jth component of $\mathcal{A}_n$, i.e. $d_m = 1$ if $n = 2m$ and $d_j = 2$ in all other cases. Then for $x = (x_1, \dots, x_m) \in \mathcal{A}_n$ define

$$||x|| := \sqrt{\frac{1}{n-1} \sum_{j=1}^{m} d_j |x_j|^2}. \tag{4}$$

Since we will in general have zero divisors, this norm cannot be multiplicative, but at least we have that $||1|| = 1$ because the sum of the d_j is $\dim \mathcal{A}_n = n-1$. See ERRATA We can use the isomorphism Ψ to define a norm on $\mathbf{P}_n$, also denoted by $||\cdot||$, and of course a metric $d(x,y) := ||x-y||$. The following theorem shows that, with respect to this natural metric, the standard units of $\mathbf{P}_n$ are the vertices of a regular simplex:

Theorem 3. *For $1 \leq k \leq n$ we have $||\sigma_k 1|| = 1$. For $1 \leq k < l \leq n$ we have*

$$d(\sigma_k 1, \sigma_l 1) = \sqrt{2 + \frac{2}{n-1}}.$$

Proof. Since $\Psi(\sigma_k 1) = (\zeta^k, \zeta^{2k}, \ldots, \zeta^{mk})$ according to (3), we compute

$$||\sigma_k 1|| = ||(\zeta^k, \zeta^{2k}, \ldots, \zeta^{mk})|| = \sqrt{\frac{1}{n-1} \sum_{j=1}^{m} d_j \cdot 1} = 1$$

because $\sum_{j=1}^{m} d_j = \dim \mathcal{A}_n = n - 1$.

For the second part we note that it is sufficient to consider the case $l = n$ because all other cases differ from this merely by a factor of absolute value 1 per component. Thus we consider $\Psi(\sigma_k 1 - \sigma_n 1) = (\zeta^k - 1, \zeta^{2k} - 1, \ldots, \zeta^{mk} - 1)$. Note that $|\zeta^{jk} - 1|^2 = (\zeta^{jk} - 1)(\overline{\zeta}^{jk} - 1) = 2 - \zeta^{jk} - \overline{\zeta}^{jk}$, hence

$$\begin{aligned}
(n-1)d(\sigma_k 1, \sigma_n 1)^2 &= \sum_{j=1}^{m} d_j (2 - \zeta^{jk} - \overline{\zeta}^{jk}) \\
&= 2\sum_{j=1}^{m} d_j - \sum_{j=1}^{m} d_j (\zeta^{jk} + \overline{\zeta}^{jk}) \\
&= 2(n-1) - \sum_{j=1}^{m} d_j (\zeta^{jk} + \overline{\zeta}^{jk}) \\
&= 2(n-1) - 2\sum_{j=1}^{n-1} \zeta^{jk}.
\end{aligned}$$

The last equality holds because only one of each pair of conjugate roots occurs in the complex components and hence we actually sum over all nth roots of unity except 1, each root occurring either twice or with weight 2. We note that $\chi \colon z \mapsto z^k$ is a character on the cyclic group of nth roots of unity. As long as k is not a multiple of n, it is different from and hence othogonal to the constant character. Hence $\sum_{j=0}^{n-1} \chi(\zeta^j) = 0$ so that the calculation continues

$$(n-1)d(\sigma_k 1, \sigma_n 1)^2 = 2(n-1) - 2\sum_{j=1}^{n-1} \zeta^{jk} = 2(n-1) + 2 - 2\sum_{j=0}^{n-1} \chi(\zeta^j) = 2n$$

and therefore $d(\sigma_k 1, \sigma_n 1) = \sqrt{2 + \frac{2}{n-1}}$ as was to be shown. □

6 Some History

The isomorphisms with better known algebras as shown in theorems 1 and 2 have been suggested long ago in usenet debates, especially for $\mathbf{P}_1$ upto $\mathbf{P}_4$. After all, they are not very hard to find and especially (2) suggests itself from the very description of polysign numbers. It seems that the first description of an isomorphism $\mathbf{P}_4 \cong \mathbb{R} \oplus \mathbb{C}$ was given by R. Chapman in October 2003[3]

[3]`http://mathforum.org/kb/message.jspa?messageID=512617&tstart=0`

and even if not shown thoroughly enough initially, it seems to have been fixed quickly. My own posts on the subject date back to 2007[4] showing $\mathbf{P}_n \cong \mathbb{R}[X]/(1 + X + \cdots X^{n-1})$ and the isomorphism $\mathbf{P}_4 \cong \mathbb{R} \oplus \mathbb{C}$ in an ad hoc manner as well as later in the same thread deriving the metric on $\mathbb{R} \oplus \mathbb{C}$ that makes the tetrahedron regular (again mostly ad hoc). Others (e.g. LWALKE3 in the same 2007 thread) have noted that the polynomial/ideal construction produces e.g. an isomorphism $\mathbf{P}_5 \cong \mathbb{C}^2$ (which I ignorantly rejected). In fact it has been stated repeatedly that $\mathbf{P}_n$ is always isomorphic to a product of $\mathbb{R}$'s and $\mathbb{C}$'s. A proof of $\mathbf{P}_3 \cong \mathbb{C}$ can also be found on T. GOLDEN's website.[5]

It is likely that the issue has come up several times in the years 2004–2009, and if so probably with the same results as presented in this paper. The usefullness of polysign numbers beyond the status of an algebraic puzzle has yet to be established. Nevertheless, in early 2009, T. GOLDEN offered a 50$ prize for the "first clean isometric isomorphism between $\mathbf{P}_4$ and $\mathbb{R} \times \mathbb{C}$".[6] This is a bit of a surprise in the light of all the different presentations of isomorphisms over the past years. I took that incident as a motivation to prepare this paper with self-contained proofs of theorems 1 and 2 as well as the metric considerations of theorem 3.

Should this paper convince T. GOLDEN to pay out that 50$ prize, I suggest to him to consider giving the money to any of the named or maybe other not named people having described the isomorphism before.

Errata

The following minor corrections had to be incorporated after first publication:

- page 2, statement of theorem 2: "*a primitve root of unity*" read "*a primitve nth root of unity*"
- page 4, two lines below equation (4): "the sum of the d_j is 1" read "the sum of the d_j is $\dim \mathcal{A}_n = n - 1$"

[4] http://groups.google.com/group/sci.math/tree/browse_frm/thread/2e2206fb5d1af68d/f545f48a6e9d84a7

[5] http://bandtechnology.com/PolySigned/ThreeSignedComplexProof.html

[6] http://groups.google.com/group/sci.math/browse_frm/thread/a30e642c0bcbdba1/114d79cc0f65764c