How to Build Triangles from Integers

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Given n rods of length $1, 2, \ldots, n$, which triangles with integer side lengths a, b, c can be built from them? This problem has been discussed in usenet.[1] The problem can be formulated more generally so that the task is to find disjoint subsets of $\{1, 2, \ldots, n\}$ such that each of these subsets sums to a given number.

For a non-negative integer n, let $N_n = \{1, 2, \ldots, n\} = \{x \in \mathbb{N} \mid x \leq n\}$. A partitioning (A, B, C, D) of N_n into four subsets is called a solution to the *problem* $(n; a, b, c, d)$ if

$$a = \sum_{x \in A} x,\ b = \sum_{x \in B} x,\ c = \sum_{x \in C} x \text{ and } d = \sum_{x \in D} x.$$

We call a problem $(n; a, b, c, d)$ *solvable* if a solution for it exists. Of course, a solution can only exist if $a + b + c + d = \frac{n(n+1)}{2}$, in which case we call the problem *valid.* It is also clear that permuting a, b, c, d does not affect validity or solveability of a problem.

For $n \in \mathbb{N}$, let $A(n)$ denote the number of solveable problems (up to permutation) for the given n, i.e. the number of solveable problems $(n; a, b, c, d)$ with $a \geq b \geq c \geq d \geq 0$. Let $B(n)$ denote the number of non-degenerate triangles that can be put together using *some* rods $\in \{1, 2, \ldots, n\}$, that is the number of integer triples (a, b, c) with $b + c > a \geq b \geq c > 0$ such that problem $(n; a, b, c, d)$ is solveable for suitable d; note that $d \leq c$ is not required and also that one problem may give rise to several triangles. Let $C(n)$ denote the number of such triangles obtainable by using *all* these rods, that is the number of triples (a, b, c) with $b + c \geq a \geq b \geq c > 0$ such that problem $(n; a, b, c, 0)$ is solveable.

Our goal is to describe these sequences, especially $B(n)$ that was originally asked for.[2] It will turn out that the sequences can be described by the following well-known sequences: For $n, k \in \mathbb{N}$, let $P(n, k)$ denote the number of partitions of n into at most k parts (or equivalently: the number of partitions of n with parts $\in N_k$).[3] Let $T(n)$ denote the number of triangles with perimeter n and integer sides.[4]

The following theorem describes all solveable problems and the results about problems involving the triangle inequality come as corollaries to it.

[1] Cf. [2]. The user with nicname(s) AI/Vikram was inspired to this question by [1, sequence A002623]. James Waldby/Calumnist, rmilka/Ross and the author contributed to that thread with partial results, which this article wants to extend.

[2] Meanwhile, sequences $A(n), B(n), C(n)$ have been included in [1] as A160438, A160456 and A160455.

[3] The case needed her specifically, namely $P(n, 4)$, is [1, sequence A001400].

[4] This is [1, sequence A005044]. It is well-known that $\sum_{i=1}^{n} T(i) = P(n-3, 4)$.

Theorem 1. *A valid problem* $(n; a, b, c, d)$ *is solveable unless it is (up to permutation of* a, b, c, d*) one of the singular exceptions* $(5; 6, 6, 2, 1)$*,* $(6; 8, 8, 3, 2)$*,* $(6; 8, 7, 3, 3)$*,* $(7; 10, 10, 4, 4)$ *or* $(7; 14, 8, 3, 3)$ *or matches (up to permutation of* a, b, c, d*) one of the patterns* $(\star; 1, 1, \star, \star)$*,* $(\star; 2, 2, \star, \star)$*,* $(\star; 3, 3, 1, \star)$*,* $(\star; 3, 3, 2, \star)$*,* $(\star; 3, 3, 3, \star)$*,* $(\star; 4, 4, 1, \star)$*,* $(\star; 4, 4, 3, \star)$ *or* $(\star; 4, 4, 4, \star)$*.*

Proof. First, we show that the listed exceptions are indeed not solveable. Assume that (A, B, C, D) is a solution for one of the exceptions.

$(\star; 1, 1, \star, \star)$: The set $\{1\}$ is the only one with element sum 1, hence $A = B = \{1\}$, contradiction.

$(\star; 2, 2, \star, \star)$: Similarly, $\{2\}$ is the only one with element sum 2, hence $A = B = \{2\}$, contradiction.

$(\star; 3, 3, 1, \star)$, $(\star; 3, 3, 2, \star)$, $(\star; 3, 3, 3, \star)$: There are exactly two possible sets with element sum 3, namely $\{3\}$ and $\{1, 2\}$, hence $A \cup B = \{1, 2, 3\}$, which leaves no possibility for C.

$(\star; 4, 4, 1, \star)$, $(\star; 4, 4, 3, \star)$, $(\star; 4, 4, 4, \star)$: There are also exactly two possible sets with element sum 4, namely $\{4\}$ and $\{1, 3\}$, hence $A \cup B = \{1, 3, 4\}$, which leaves no possibility for C.

$(5; 6, 6, 2, 1)$: We must have $|A| \geq 2$, $|B| \geq 2$. Together with $|C| \geq 1$, $|D| \geq 1$, we arrive at the contradiction $5 \geq 6$.

$(6; 8, 8, 3, 2)$: We have $6 \notin C \cup D$, hence wlog. $6 \in A$, which implies $A = \{2, 6\}$. But necessarily $D = \{2\}$, contradiction

$(6; 8, 7, 3, 3)$: We have $|A|, |B| \geq 2$ and $|C|, |D| \geq 1$, hence necessarily $|C| = |D| = 1$, i.e. $C = D = \{3\}$, contradiction.

$(7; 10, 10, 4, 4)$: As noted under the forth item, $C \cup D = \{1, 3, 4\}$. From $|A|, |B| \geq 2$ we then conclude $|A| = |B| = 2$ and hence $\min A, \min B \geq 10 - 7 = 3$, but one set must contain 2, contradiction.

$(7; 14, 8, 3, 3)$: We must have $C \cup D = \{1, 2, 3\}$ and as before $|A| = |B| = 2$, but 14 cannot be obtained from adding two different numbers ≤ 7.

Secondly, to show that conversely all non-exceptional problems are solveable, use induction on n: The case $n = 0$ is trivial. Assume $n > 0$ and let $(n; a, b, c, d)$ be a valid problem that is not among the listed exceptions. Assume without loss of generality that $a \geq b \geq c \geq d$. If we have a solution (A, B, C, D) for problem $(n - 1; a - n, b, c, d)$, then clearly$(A \cup \{n\}, B, C, D)$ is a solution for $(n; a, b, c, d)$. This greedy approach can fail for two reasons: We may have $a < n$ or the problem $(n - 1; a - n, b, c, d)$ may belong to the list of exception problems without solution.

If $a < n$, then $\frac{n(n+1)}{2} = a + b + c + d \leq 4n - 4$ implies $2 \leq n \leq 6$ and this leaves only the following few posibilities with $n > a \geq b \geq c \geq d$: $(2; \mathbf{1}, \mathbf{1}, 1, 0)$, $(3; \mathbf{2}, \mathbf{2}, 2, 0)$, $(3; \mathbf{2}, \mathbf{2}, 1, 1)$, $(4; \mathbf{3}, \mathbf{3}, \mathbf{3}, 1)$, $(4; 3, 3, \mathbf{2}, \mathbf{2})$, $(5; \mathbf{4}, \mathbf{4}, \mathbf{4}, 3)$. These all match (at least) one of the exception patterns, as indicated by highlighting the numbers making up the pattern in boldface. We may therefore assume for the rest of the argument that $a \geq n$ and it is only necessary to investigate all cases where $(n-1; a-n, b, c, d)$ is listed as an exception and then to give an alternate solution for these cases.

All problems $(n; a, b, c, d)$ such that $(n-1; a-n, b, c, d)$ is (up to permutation) one of the singular exceptions $(m; x, y, z, w)$ are readily obtained by adding $m+1$ to all components in turn (and possibly resorting) as done in table 1. One verifies that all entries in the right column of table 1 are listed in table 3 with an explicit solution.

If $(n-1; a-n, b, c, d)$ matches (up to permutation) a pattern of the form $(\star; x, x, \star, \star)$, it is clear that $a-n=x$ because otherwise $(n; a, b, c, d)$ would match the same exception pattern. It follows that $\frac{n(n+1)}{2} = a+b+c+d \leq 3n+4x$. Note that $\frac{n(n+1)}{2} \leq 3n+11$ implies $n \leq 7$ and $\frac{n(n+1)}{2} \leq 3n+6$ implies $n \leq 6$, thus only a limited number of candidates have to be checked. Likewise, for patterns of the form $(\star; x, x, y, \star)$ with $x \geq y$, we must have $a-n \in \{x, y\}$ and $\frac{n(n+1)}{2} \leq 2n+3x+y$. Note that $\frac{n(n+1)}{2} \leq 2n+19$ implies $n \leq 7$ and $\frac{n(n+1)}{2} \leq 2n+13$ implies $n \leq 6$. Also, the forth number $\frac{n(n+1)}{2} - (2x+y+n)$ must be $\leq a$. This makes it an easy task to list all problems leading to exceptional patterns, the result of which is shown in table 2. Again, one verifies that all entries in the right column of table 2 are either exceptions themselves–as indicated by boldface components–or are listed in table 3 with an explicit solution.

Since all cases that lead to an exception are either exceptions themselves or are explicitly solved in table 3, the theorem is proved. $\square$

Table 1: Singular exceptions and what leads to them by a greedy step.

$(5; 6, 6, 2, 1)$	$(6; 12, 6, 2, 1)$, $(6; 8, 6, 6, 1)$, $(6; 7, 6, 6, 2)$
$(6; 8, 8, 3, 2)$	$(7; 15, 8, 3, 2)$, $(7; 10, 8, 8, 2)$, $(7; 9, 8, 8, 3)$
$(6; 8, 7, 3, 3)$	$(7; 15, 7, 3, 3)$, $(7; 14, 8, 3, 3)$, $(7; 10, 8, 7, 3)$
$(7; 14, 8, 3, 3)$	$(8; 22, 8, 3, 3)$, $(8; 16, 14, 3, 3)$, $(8; 14, 11, 8, 3)$
$(7; 10, 10, 4, 4)$	$(8; 18, 10, 4, 4)$, $(8; 12, 10, 10, 4)$

Table 2: Exception patterns and what leads to them by a greedy step.

$(\star; 1, 1, \star, \star)$	$(6; 7, 7, 6, 1)$, $(\mathbf{5}; \mathbf{6}, \mathbf{6}, \mathbf{2}, \mathbf{1})$, $(5; 6, 5, 3, 1)$, $(5; 6, \mathbf{4}, \mathbf{4}, \mathbf{1})$, $(4; 5, 4, 1, 0)$, $(4; 5, 3, \mathbf{1}, \mathbf{1})$, $(4; 5, \mathbf{2}, \mathbf{2}, 1)$, $(3; 4, \mathbf{1}, \mathbf{1}, 0)$
$(\star; 2, 2, \star, \star)$	$(7; 9, 9, 8, 2)$, $(\mathbf{6}; \mathbf{8}, \mathbf{8}, \mathbf{3}, \mathbf{2})$, $(6; 8, 7, 4, 2)$, $(6; 8, 6, 5, 2)$, $(5; 7, 6, 2, 0)$, $(5; 7, 5, 2, 1)$, $(5; 7, 4, \mathbf{2}, \mathbf{2})$, $(5; 7, \mathbf{3}, \mathbf{3}, \mathbf{2})$, $(4; 6, \mathbf{2}, \mathbf{2}, 0)$, $(4; 6, 2, \mathbf{1}, \mathbf{1})$
$(\star; 3, 3, 1, \star)$	$(6; 9, 8, 3, 1)$, $(5; 8, \mathbf{3}, \mathbf{3}, \mathbf{1})$, $(5; 6, \mathbf{3}, \mathbf{3}, \mathbf{3})$
$(\star; 3, 3, 2, \star)$	$(6; 9, 7, 3, 2)$, $(\mathbf{6}; \mathbf{8}, \mathbf{7}, \mathbf{3}, \mathbf{3})$, $(5; 8, 3, \mathbf{2}, \mathbf{2})$, $(5; 7, \mathbf{3}, \mathbf{3}, \mathbf{2})$
$(\star; 3, 3, 3, \star)$	$(6; 9, 6, 3, 3)$, $(5; 8, \mathbf{3}, \mathbf{3}, \mathbf{1})$
$(\star; 4, 4, 1, \star)$	$(6; 10, 6, 4, 1)$, $(6; 7, 6, 4, 4)$, $(5; 9, 4, \mathbf{1}, \mathbf{1})$, $(5; 6, \mathbf{4}, \mathbf{4}, \mathbf{1})$
$(\star; 4, 4, 3, \star)$	$(7; 11, 10, 4, 3)$, $(\mathbf{7}; \mathbf{10}, \mathbf{10}, \mathbf{4}, \mathbf{4})$, $(6; 10, \mathbf{4}, \mathbf{4}, \mathbf{3})$, $(6; 9, \mathbf{4}, \mathbf{4}, \mathbf{4})$
$(\star; 4, 4, 4, \star)$	$(7; 11, 9, 4, 4)$, $(6; 10, \mathbf{4}, \mathbf{4}, \mathbf{3})$

Table 3: Some special problems with non-greedy solutions

(8;22,8,3,3)	({4,5,6,7}, {8}. {3}, {1,2})
(8;18,10,4,4)	({5,6,7}, {2,8}. {4}, {1,3})
(8;18,8,8,2)	({3,4,5,6}, {8}. {1,7}, {2})
(8;16,14,3,3)	({4,5,7}, {6,8}. {3}, {1,2})
(8;16,10,8,2)	({1,4,5,6}, {3,7}. {8}, {2})
(8;14,11,8,3)	({1,2,5,6}, {4,7}. {8}, {3})
(8;12,10,10,4)	({1,5,6}, {2,8}. {3,7}, {4})
(8;10,10,8,8)	({1,4,5}, {3,7}. {8}, {2,6})
(7;15,8,3,2)	({4,5,6}, {1,7}. {3}, {2})
(7;15,7,3,3)	({4,5,6}, {7}. {3}, {1,2})
(7;11,9,4,4)	({5,6}, {2,7}. {4}, {1,3})
(7;11,10,4,3)	({5,6}, {1,2,7}. {4}, {3})
(7;10,8,8,2)	({4,6}, {3,5}. {1,7}, {2})
(7;10,8,7,3)	({1,4,5}, {2,6}. {7}, {3})
(7;9,9,8,2)	({4,5}, {3,6}. {1,7}, {2})
(7;9,8,8,3)	({4,5}, {1,7}. {2,6}, {3})
(6;12,6,2,1)	({3,4,5}, {6}. {2}, {1})
(6;8,6,6,1)	({3,5}, {6}. {2,4}, {1})
(6;7,6,6,2)	({3,4}, {6}. {1,5}, {2})
(6;7,6,4,4)	({2,5}, {6}. {4}, {1,3})
(6;10,6,4,1)	({2,3,5}, {6}. {4}, {1})
(6;9,6,3,3)	({4,5}, {6}. {3}, {1,2})
(6;9,7,3,2)	({4,5}, {1,6}. {3}, {2})
(6;9,8,3,1)	({4,5}, {2,6}. {3}, {1})
(6;8,7,4,2)	({3,5}, {1,6}. {4}, {2})
(6;8,6,5,2)	({1,7}, {6}. {5}, {2})
(6;7,7,6,1)	({3,4}, {2,5}. {6}, {1})
(5;7,5,2,1)	({3,4}, {5}. {2}, {1})
(5;7,6,2,0)	({3,4}, {1,5}. {2}, {})
(5;6,5,3,1)	({2,4}, {5}. {3}, {1})
(4;5,4,1,0)	({2,3}, {4}. {1}, {})

Corollary 1. *For $n \geq 8$, we have*

$$P\left(\frac{n(n+1)}{2}, 4\right) - A(n) = 2\left[\frac{n(n+1)}{4}\right] + 5.$$

Proof. For $n \geq 8$, only the patterns of the form $(\star; x, x, \star, \star)$ and $(\star; x, x, y, \star)$ of theorem 1 play a role and no two of them can match the same problem. Clearly, $(\star; x, x, y, \star)$ matches if and only if the forth number is $\frac{n(n+1)}{2} - 2x - y$, thus each such pattern contributes 1 to the difference $P(\frac{n(n+1)}{2}, 4) - A(n)$. For patterns $(\star; x, x, \star, \star)$, the remaining two numbers $u \leq v$ must add up to $t := \frac{n(n+1)}{2} - 2x$, which is possible in $P(t, 2) = \left[\frac{t}{2}\right] + 1$ ways. Therefore $(\star; 1, 1, \star, \star)$ and $(\star; 2, 2, \star, \star)$ contribute

$$\left[\frac{\frac{n(n+1)}{2} - 2}{2}\right] + 1 + \left[\frac{\frac{n(n+1)}{2} - 4}{2}\right] + 1 = 2\left[\frac{n(n+1)}{4}\right] - 1.$$

The claim follows by adding the contribution of the six patterns of the form $(\star; x, x, y, \star)$. □

Corollary 2. *If $n \geq 8$ and $r \in \{0, 1, 2, 3\}$ is the remainder of n mod 4 then*

$$P\left(\frac{n(n+1)}{2} - 3, 4\right) - B(n) = \begin{cases} 11 & \text{if } r = 1 \text{ or } r = 2, \\ 12 & \text{if } r = 0 \text{ or } r = 3. \end{cases}$$

Proof. Once again, the singular exceptions play no role if $n \geq 8$.

If we assume that an exception pattern $(\star; x, x, y, \star)$ with $x \geq y$ eliminates a triangle then either x, x, y are the sides of this triangle or the third side of the triangle is the forth number $\frac{n(n+1)}{2} - 2x - y$ and must be less than $2x \leq 8$. For $n \geq 8$, only the first possibility remains, hence the patterns $(\star; x, x, y, \star)$ together contribute 6 to the difference $P\left(\frac{n(n+1)}{2} - 3, 4\right) - B(n)$.

For pattern $(\star; 1, 1, \star, \star)$, clearly the triangle with side lengths $(1, 1, 1)$ is eliminated. The only other possibility is triangle $(a, a, 1)$ with $2a + 2 = \frac{n(n+1)}{2}$. Thus $(\star; 1, 1, \star, \star)$ contributes 2 if $\frac{n(n+1)}{2}$ is even (i.e. if r is 0 or 3) and 1 else.

For pattern $(\star; 2, 2, \star, \star)$, it is clear that triangles $(2, 2, 1)$, $(2, 2, 2)$ and $(2, 2, 3)$ are eliminated. The only other possibilities are $(a, b, 2)$ with $a = b$ or $a = b + 1$ and $a + b + 4 = \frac{n(n+1)}{2}$. Hence $(\star; 2, 2, \star, \star)$ always contribute 4 to the difference. □

Corollary 3. *If $n \geq 4$ then*

$$C(n) = T\left(\frac{n(n+1)}{2}\right).$$

Proof. For triangles using all rods, only solutions to problems of the form $(n; a, b, c, 0)$ need to be considered. This immediately rules out all singular exceptions and also the patterns $(\star; x, x, y, \star)$ because $2x + y \in \{7, 8, 9, 11, 12\}$ cannot be of the form $\frac{n(n+1)}{2}$. For patterns $(\star; x, x, \star, \star)$ with $x \leq 2$ the only problems $(n; a, x, x, 0)$ that correspond to triangles, i.e. those that have $1 \leq a < 2x$, occur for $\frac{n(n+1)}{2} < 4x \leq 8$, hence $n \leq 3$. □

As a consequence of the above corollaries, the sequences $A(n), B(n), C(n)$ are easily computed from other known sequences, once a few initial terms have been determined explicitly. Some small result are presented in table 4 with deviations from the corollary formulae indicated by italics.

Table 4: Initial terms of sequences $A(n), B(n), C(n)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$A(n)$	*1*	*1*	*2*	*5*	*13*	35	*93*	*215*	437	815	1436	2413	3886
$B(n)$	*0*	*0*	*0*	*0*	*3*	*20*	*70*	*172*	366	709	1274	2166	3537
$C(n)$	0	0	*0*	*0*	2	7	12	16	27	48	70	91	127

References

[1] N. J. A. Sloane, Ed. (2008), *The On-Line Encyclopedia of Integer Sequences,* `http://www.research.att.com/~njas/sequences/`

[2] Usenet thread, *Number of Triangles from given set of rods,* 2009-05-04, available at `http://groups.google.com/group/sci.math/browse_frm/thread/70c1521d9143d634`, also at `http://mathforum.org/kb/message.jspa?messageID=6695991`